1. (i) Find and classify the critical points of the function \( f(x, y) = y^2 - 2xy - x^3 + 5x \). By considering the behaviour of \( f(x, y) \) on the axes, show that there is no global maximum or minimum. (13 marks)

(ii) Use the method of Lagrange multipliers to find the maximum and minimum values of \( 5x \) subject to the constraint \( x^2 + xy + y^2 = 3x + 3y + 9 \). (12 marks)

2. Consider the following region \( D \).

   (i) What are the equations of the three edges of the region? (3 marks)

   (ii) Consider the integral

   \[
   I = \iint_D y^{-1} \sin\left(\frac{1}{3} \pi y\right) \, dA
   \]

   Work out the limits to give two different expressions for \( I \), one as an integral of the form \( \int_{x=\ldots}^{\ldots} \int_{y=\ldots}^{\ldots} y^{-1} \sin\left(\frac{1}{3} \pi y\right) \, dy \, dx \), and the other of the form \( \int_{y=\ldots}^{\ldots} \int_{x=\ldots}^{\ldots} y^{-1} \sin\left(\frac{1}{3} \pi y\right) \, dx \, dy \). (6 marks)

   (iii) Use the second expression to evaluate \( I \). (5 marks)

   (iv) Evaluate \( J = \iint_D (1 - y)/x \, dA \). (6 marks)
3 (i) Consider the vector field
\[ \mathbf{u} = (\sin(3x + 4y), \sin(3x + 4y), 2\cos(3x + 4y)). \]
Calculate the following:
(a) \( \text{div}(\mathbf{u}) \)
(b) \( \text{grad}(\text{div}(\mathbf{u})) \)
(c) \( \text{curl}(\mathbf{u}) \)
(d) \( \text{curl}(\text{curl}(\mathbf{u})) \)
(e) \( \nabla^2(\mathbf{u}). \)
Verify that \( \text{curl}(\text{curl}(\mathbf{u})) = \text{grad}(\text{div}(\mathbf{u})) - \nabla^2(\mathbf{u}). \)  
(16 marks)

(ii) Consider the scalar field \( f(x, y, z) = e^{-(x^2+z^2)/2}. \) Find \( \nabla(f) \) and \( \nabla^2(f). \) Give a geometric description of the points where \( \nabla^2(f) = 0. \)
(9 marks)

4 (i) Let \( S \) be the surface given by \( z = x^2 + y^2 \) with \( 0 \leq z \leq 4, \) and let \( C \) be the boundary of \( S. \) Consider the vector field \( \mathbf{F} = x\mathbf{i} + 2x\mathbf{j} + 3z\mathbf{k}. \) Evaluate the integrals \( \iint_{S} (\nabla \times \mathbf{F}).d\mathbf{A} \) and \( \int_{C} \mathbf{F}.d\mathbf{r} \) separately, and check that they are the same (in accordance with Stokes's Theorem).
(18 marks)

(ii) Let \( E \) be the spherical ball of radius one centred at the origin, and let \( T \) be the boundary of \( E. \) Let \( \mathbf{G} \) be the vector field \( (z, 0, z). \) Evaluate the integrals \( \iiint_{E} \nabla \cdot \mathbf{G} \, dV \) and \( \iint_{T} \mathbf{G}.d\mathbf{A} \) separately, and check that they are the same (in accordance with the Divergence Theorem).
(12 marks)

You may use the identities
\[ \sin(\alpha)\cos^2(\alpha) = \frac{1}{4}(\sin(\alpha) + \sin(3\alpha)) \]
\[ \sin^2(\alpha)\cos(\alpha) = \frac{1}{4}(\cos(\alpha) - \cos(3\alpha)). \]

End of Question Paper
Formula Sheet for MAS243

Trigonometry

\[
\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B
\]

\[
\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B
\]

\[
\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}
\]

\[
a \cos \theta + b \sin \theta = R \cos(\theta - \alpha)\text{ where } R = \sqrt{a^2 + b^2}\text{ and } \cos \alpha = \frac{a}{R}, \sin \alpha = \frac{b}{R}
\]

\[
\cos^2 \theta = \frac{1}{2} (\cos 2\theta + 1)
\]

\[
\cos^3 \theta = \frac{1}{4} (3 \cos \theta + \cos 3\theta)
\]

\[
\cos^4 \theta = \frac{1}{8} (3 + 4 \cos 2\theta + \cos 4\theta)
\]

\[
\sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta)
\]

\[
\sin^3 \theta = \frac{1}{4} (3 \sin \theta - \sin 3\theta)
\]

\[
\sin^4 \theta = \frac{1}{8} (3 - 4 \cos 2\theta + \cos 4\theta)
\]