Data provided: Formula Sheet

The University Of Sheffield.

# MAS243

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring Semester 2012–2013**

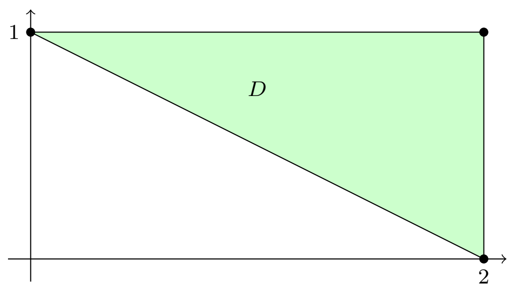
**Mathematics IV (Electrical)** **2 hours**

*Attempt all FOUR questions. Note that there are different numbers of marks for different questions.*

**1** (i) Find and classify the critical points of the function $f(x, y) = y^2 - 2xy - x^3 + 5x$. By considering the behaviour of $f(x, y)$ on the axes, show that there is no global maximum or minimum. ***(13 marks)***

(ii) Use the method of Lagrange multipliers to find the maximum and minimum values of $5x$ subject to the constraint $x^2 + xy + y^2 = 3x + 3y + 9$. ***(12 marks)***

**2** Consider the following region $D$.




(i) What are the equations of the three edges of the region? ***(3 marks)***

(ii) Consider the integral

$$I = \iint_D y^{-1} \sin(\tfrac{1}{3}\pi y)\, dA$$

Work out the limits to give two different expressions for $I$, one as an integral of the form $\int_{x=\cdots}^{\cdots} \int_{y=\cdots}^{\cdots} y^{-1} \sin(\frac{1}{3}\pi y)\, dy\, dx$, and the other of the form $\int_{y=\cdots}^{\cdots} \int_{x=\cdots}^{\cdots} y^{-1} \sin(\frac{1}{3}\pi y)\, dx\, dy$. ***(6 marks)***

(iii) Use the second expression to evaluate $I$. ***(5 marks)***

(iv) Evaluate $J = \iint_D (1 - y)/x\, dA$. ***(6 marks)***

**3** (i) Consider the vector field

$$\mathbf{u} = (\sin(3x+4y),\ \sin(3x+4y),\ 2\cos(3x+4y)).$$

Calculate the following:

(a) $\text{div}(\mathbf{u})$

(b) $\text{grad}(\text{div}(\mathbf{u}))$

(c) $\text{curl}(\mathbf{u})$

(d) $\text{curl}(\text{curl}(\mathbf{u}))$

(e) $\nabla^2(\mathbf{u})$.

Verify that $\text{curl}(\text{curl}(\mathbf{u})) = \text{grad}(\text{div}(\mathbf{u})) - \nabla^2(\mathbf{u})$. ***(16 marks)***

(ii) Consider the scalar field $f(x,y,z) = e^{-(x^2+z^2)/2}$. Find $\nabla(f)$ and $\nabla^2(f)$. Give a geometric description of the points where $\nabla^2(f) = 0$. ***(9 marks)***

**4** (i) Let $S$ be the surface given by $z = x^2 + y^2$ with $0 \leq z \leq 4$, and let $C$ be the boundary of $S$. Consider the vector field $\mathbf{F} = x\,\mathbf{i} + 2x\,\mathbf{j} + 3x\,\mathbf{k}$. Evaluate the integrals $\iint_S (\nabla \times \mathbf{F}).d\mathbf{A}$ and $\int_C \mathbf{F}.d\mathbf{r}$ separately, and check that they are the same (in accordance with Stokes's Theorem). ***(18 marks)***

(ii) Let $E$ be the spherical ball of radius one centred at the origin, and let $T$ be the boundary of $E$. Let $\mathbf{G}$ be the vector field $(z, 0, z)$. Evaluate the integrals $\iiint_E \nabla.\mathbf{G}\,dV$ and $\iint_T \mathbf{G}.d\mathbf{A}$ separately, and check that they are the same (in accordance with the Divergence Theorem). ***(12 marks)***

You may use the identities

$$\sin(\alpha)\cos^2(\alpha) = \frac{1}{4}(\sin(\alpha) + \sin(3\alpha))$$
$$\sin^2(\alpha)\cos(\alpha) = \frac{1}{4}(\cos(\alpha) - \cos(3\alpha)).$$

## End of Question Paper

**Formula Sheet for MAS243**
**Trigonometry**

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$a \cos\theta + b \sin\theta = R\cos(\theta - \alpha) \text{ where } R = \sqrt{a^2 + b^2} \text{ and } \cos\alpha = \frac{a}{R},\ \sin\alpha = \frac{b}{R}$$

$$\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$$

$$\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$$

$$\cos^4\theta = \frac{1}{8}(3 + 4\cos 2\theta + \cos 4\theta)$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$$

$$\sin^4\theta = \frac{1}{8}(3 - 4\cos 2\theta + \cos 4\theta)$$