Scattering of gravitational waves by a neutron star

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Overview

1. Perturbation theory in general relativity
2. Polytropic stars
3. Plane waves and scattering
4. Scattering cross sections
5. Complex angular momentum techniques
6. Conclusions
Motivation

Direct detection of GWs emanating from:

- Black hole binary mergers (10 detections in GWTC-1).
- Neutron star mergers (1 detection in GWTC-1).
- Neutron star black hole mergers.

How do GWs propagate through curved space?

Is secondary scattering important / detectable?
Wavefront as a congruence of geodesics incident on a constant density star.
Caustics
Toy model

- We model time independent scattering of a plane wave by a compact body.
- Spherically symmetric polytropic star, radius $R$, mass $M$, polytropic index $n$. Metric

$$ds^2 = -f(r)dt^2 + h(r)^{-1}dr^2 + r^2d\Omega^2.$$  \hspace{1cm} (1)

When $r > R$, $f = h = 1 - 2M/r$ (Birkhoff’s theorem).
- The progenitor neutron stars from the binary merger detected by the LIGO collaboration: $4.4 \lesssim c^2R/(GM) \lesssim 7.7$ with 90% credibility.

<table>
<thead>
<tr>
<th></th>
<th>Sun</th>
<th>White dwarf</th>
<th>Neutron star</th>
<th>UCO</th>
<th>Buchdahl</th>
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</thead>
<tbody>
<tr>
<td>$c^2R/\sqrt{GM}$</td>
<td>$4.7 \times 10^5$</td>
<td>$\sim 10^3 - 10^4$</td>
<td>$\sim 6$</td>
<td>$&lt; 3$</td>
<td>$&gt; 9/4$</td>
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Table: Tenuities
Geodesic deflection function

- $R/M > 3$: minimum in $\Theta(b) \rightarrow$ rainbow scattering.
- $R/M \approx 3.5$: glory
- $R/M \leq 3$: divergence in $\Theta(b) \rightarrow$ orbiting.
Perturbation theory

- Perturb a known solution

\[ g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}. \]  \hspace{1cm} (2)

- Linearise Einstein’s field equations

\[ \delta G_{\mu\nu}(g^0_{\mu\nu}, h_{\mu\nu}) = \frac{8\pi G}{c^4} \delta T_{\mu\nu}(g^0_{\mu\nu}, h_{\mu\nu}). \]  \hspace{1cm} (3)

- For a spherically symmetric background, eq. (3) separates into tensor spherical harmonics, classified as axial (-) or polar (+) according to their transformation properties under the parity operation \( \theta \rightarrow \pi - \theta \), \( \phi \rightarrow \phi + \pi \).
The equations for each parity sector decouple.

By Fourier transforming, Regge and Wheeler (1957) and Zerilli (1970) reduced the axial (−) and polar (+) radial equations to

\[
\left[ \frac{d^2}{dr_+^2} + \omega^2 - V_l^\pm(r) \right] \Phi_{lm}^\pm(r, \omega) = 0. \tag{4}
\]

Martel and Poisson (2005) show how to construct the radiative part of the metric in the far field \((r \to \infty)\),

\[
h_{AB}^{\text{rad.}} \sim r \int_{-\infty}^{\infty} \sum_{l,m,p} \left[ A_{lp}^{\text{in}}(\omega) e^{-i\omega r_+} + A_{lp}^{\text{out}}(\omega) e^{i\omega r_+} \right] e^{-i\omega t} Y_{lmp} \, d\omega. \tag{5}
\]

Alternative is to study perturbations of the Weyl scalars within Newman-Penrose formalism.
Perturbations of a relativistic star

- Developed in analogy with Newtonian approach by Thorne et al (1967...). Focus on polar modes.

- Chandrasekhar and Ferrari (1991....) reformulated the theory of stellar oscillations following black hole perturbation theory. Interpret as scattering problem.

- Axial:

\[
\left[\frac{d^2}{dr_*^2} + \omega^2 - V_l(r)\right] \tilde{Q}_{lm} = 0.
\]  

(6)

where \( dr/dr_* = \sqrt{f h} \).

- Equation (6) reduces to Regge-Wheeler equation for \( \Phi_{lm}^- \) in the exterior \( (r > R) \).
There is a discontinuity in $\mathcal{V}_l$ at the surface of stars of constant density ($n = 0$), or those with a solid crust. This plays an important role in the quasinormal mode spectrum (Zhang et al 2011).
Perturbations of a relativistic star - the polar sector

- The polar sector is determined by the coupled ODEs (Lindblom and Detweiler 1983)

\[
\left[ \frac{d^2}{dr_*^2} + \omega^2 \right] \tilde{S}_{lm} + L_1(\tilde{S}_{lm}, \tilde{F}_{lm}, l) = 0 \tag{8}
\]

\[
\left[ \frac{d^2}{dr_*^2} + \frac{\omega^2}{c_s^2} \right] \tilde{F}_{lm} + L_2(\tilde{S}_{lm}, \tilde{F}_{lm}, l) = 0 \tag{9}
\]

- Encodes interaction with fluid of star.

- Suitable junction conditions at the surface relate \( \{ \tilde{Q}, \tilde{F}, \tilde{S} \} \) and \( \Phi^\pm \).
Polytropic stellar models

- Perfect fluid
  \[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}. \]  
  \[(10)\]

- Tolman Oppenheimer Volkoff equation
  \[ \frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)}. \]  
  \[(11)\]

- Constraint
  \[ \frac{dm}{dr} = 4\pi r^2 \rho. \]  
  \[(12)\]

- Equation of state
  \[ p(\rho) = \kappa \rho^{1+1/n}. \]  
  \[(13)\]
Defining a plane wave

- Left circularly polarized GW, travelling up $z$-axis, helicity $+2$ ($\omega > 0$):

$$h_{ij}^{\text{plane}} = \text{Re}\{e^{i(2\phi - \omega(t - r_* \cos \theta))} \left[ \begin{array}{ccc}
\sin^2 \theta & r \sin \theta \cos \theta & ir \sin^2 \theta \\
0 & r^2 \cos^2 \theta & ir^2 \sin \theta \cos \theta \\
0 & 0 & -r^2 \sin^2 \theta
\end{array} \right] \right\}, \quad (14)$$

- Long range nature of field of star means wave is ‘distorted’ - tortoise coordinate $r_*$ in the exponent

- Master functions satisfy $\Phi_{l(-m)p} = \Phi^*_{lmp}$,

$$\Phi_{l2,-1}^{\text{plane}}(r, t) \sim \frac{2\pi C_{l2}}{\omega} \left((-1)^{l+1} e^{-i\omega r_*} + e^{i\omega r_*}\right) e^{-i\omega t}, \quad (15)$$

$$\Phi_{l2,+1}^{\text{plane}}(r, t) \sim -\frac{2\pi i C_{l2}}{\omega} \left((-1)^{l+1} e^{-i\omega r_*} + e^{i\omega r_*}\right) e^{-i\omega t}, \quad (16)$$
Gravitational wave energy

- Find $h_{AB}^{\text{scat}}$ by matching in the far field

$$
\Phi_{lmp}^{\text{total}} = \Phi_{lmp}^{\text{scat}} + \Phi_{lmp}^{\text{plane}},
$$

where

$$
\Phi_{lmp}^{\text{total}} \sim (A_{lp}^{\text{in}}(\omega)e^{-i\omega r} + A_{lp}^{\text{out}}(\omega)e^{i\omega r}).
$$

- A gauge invariant definition of the perturbations energy is given by Brill and Hartle by ‘space time averaging’ over many wavelengths.

$$
\left\langle \frac{dE}{dt \, d\Omega} \right\rangle = -\left\langle \lim_{r \to \infty} r^2 f T_{rt} \right\rangle = -\frac{1}{r^2} \frac{1}{32\pi} \langle \partial_r h^{AB} \partial_t h_{AB} \rangle.
$$

- Divide LHS of eq. (19) by the incident waves flux to get differential scattering cross section.
Scattering cross sections

- Scattering cross section (GW) consists of helicity conserving, $\hat{f}_2$, and helicity reversing, $\hat{g}_2$, components

$$\frac{d\sigma}{d\Omega} = \begin{cases} |\hat{f}_2|^2 + |\hat{g}_2|^2; & \text{GW} \\ |\hat{f}_0|^2; & \text{scalar} \end{cases}$$

- Scattering amplitudes with $S_{lp} \equiv (-1)^{l+1} A_{lp}^{\text{out}} / A_{lp}^{\text{in}}$:

$$\hat{f}_2(\theta) \equiv \frac{\pi}{i\omega} \sum_{l=2}^{\infty} \sum_{p=\pm1} \left( \frac{2l+1}{4\pi} \right)^{1/2} (S_{lp}(\omega) - 1)_{-2} Y^{l2}(\theta),$$

$$\hat{g}_2(\theta) \equiv \frac{\pi}{i\omega} \sum_{l=2}^{\infty} \sum_{p=\pm1} p \left( \frac{2l+1}{4\pi} \right)^{1/2} (S_{lp}(\omega) - 1)_{2} Y^{l2}(\theta),$$

$$\hat{f}_0(\theta) \equiv \frac{\pi}{i\omega} \sum_{l=0}^{\infty} \left( \frac{2l+1}{4\pi} \right)^{1/2} (S_{l}(\omega) - 1)_{0} Y^{l0}(\theta)$$
Low frequency gravitational waves

Figure: Helicity preserving cross section for GW scattering by a polytropic star (solid) and helicity reversing (dot-dashed). Black hole low frequency approximations (red dashed), $M\omega \ll 1$. 
Mid frequency gravitational waves

\[ M\omega \sim 1 \]

\[ M\omega = 4 \]

\[ R/M = 6 \]

\[ M^{-2} d\sigma/d\Omega \]

\[ \theta \]

\[ \text{GW scalar} \]

\[ \text{GW helicity preserving} \]

\[ \text{GW helicity reversing} \]

**Figure**: Rainbow scattering by a polytropic star.
Rainbow scattering

\[ R/M = 6 \]
\[ M\omega = 1 \]
\[ M^{-2} \frac{d\sigma}{d\Omega} \]
\[ \theta \]

GW scalar
GW helicity preserving
GW helicity reversing

\[ R/M = 6 \]
\[ M\omega = 2 \]
\[ M^{-2} \frac{d\sigma}{d\Omega} \]
\[ \theta \]

\[ R/M = 6 \]
\[ M\omega = 4 \]
\[ M^{-2} \frac{d\sigma}{d\Omega} \]
\[ \theta \]

\[ R/M = 6 \]
\[ M\omega = 8 \]
\[ M^{-2} \frac{d\sigma}{d\Omega} \]
\[ \theta \]
Rainbow scattering

Figure: Rainbow scattering by different stellar models
The nuclear rainbow

Figure: Nuclear rainbow scattering [Satchler]. Allowed identification of correct (Woods-Saxon) nuclear potential model
Very compact objects

\[ \theta_r \approx 189.4^\circ \text{ for } R=3.5M. \]

Backward glory enhanced.
Near field scattering profile

Figure: Near field scattering profile for a polytropic star with $n = 1$, $R/M = 6$. Significant amplification of the wave can be seen at the cusp caustic.
Complex angular momentum

\[
\text{Im}(\lambda) \quad \text{Re}(\lambda) \quad C
\]

1/2 \quad 3/2 \quad 5/2 \quad 7/2

Sommerfeld-Watson transformation

\[
\hat{f}_0(\theta) = \sum_{\ell=0}^{+\infty} (-1)^\ell F(\ell) = \frac{i}{2} \int_C d\lambda \frac{F(\lambda - 1/2)}{\cos(\pi \lambda)},
\]

where \( \lambda \equiv \ell + 1/2 \in \mathbb{C} \) and

\[
F(\lambda - 1/2) = \frac{\lambda}{i\omega} \left[ S_{\lambda-1/2}(\omega) - 1 \right] P_{\lambda-1/2}(- \cos \theta).
\]
Regge poles for a compact star

- By deforming the contour, write \( \hat{f}_0(\theta) \) as a background integral + a sum over poles of \( F(\lambda - 1/2) \) in the plane.

- The Regge poles correspond to zeroes of \( A_{\lambda - 1/2}^{\text{in}}(\omega) \). In other words, they satisfy the purely outgoing boundary condition in the far field.
Regge pole contribution to the scattering cross section

\[
\text{Scattering cross section } \\
R = 6M \text{ and } 2M\omega = 16
\]

Exact

Sum over Regge poles (n = 2)

Sum over Regge poles (n = 20)

Sum over Regge poles (n = 42)

Sum over Regge poles (n = 81) + background integral

Scattering cross section
R = 6M and 2M\omega = 16
Regge poles for scattering by a transparent sphere

Poles can be associated with different types of ‘resonances’.

- Broad resonances (2): Penetrate deep inside the sphere.
- Surface waves (4): Graze the surface of the sphere.
- Narrow resonances (1): Resonate with trapped states.

**Figure:** Regge poles for scattering by a transparent sphere [Nussenzweig, H.M. 2006].
Regge poles for scattering by a ultra compact object

- Broad resonances: Penetrate deep inside the light ring.
- Surface waves: Graze the light ring.
- Narrow resonances: Resonate with trapped states.
Conclusions

- For moderate to high frequencies we see a rainbow interference pattern for gravitational scattering by a compact, horizon-less object.
- Known GW sources and neutron star masses correspond to $M\omega \lesssim 0.1$ (the highest frequency source being a pulsar), so we are unlikely to see this in practice.
- GW-fluid coupling has small effect.
- Enhanced backward glory for $R \lesssim 3.5M$.
- Illuminating analogies with the atmospheric rainbow and nuclear scattering.
- Further work: scattering of a chirp signal, interpretation of Regge pole spectrums, effects of rotation...